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Bipartable Graphs

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We consider a construction which associates with a graph G another graph G' such that if G' is a bipartite graph, G is perfectly orderable. For such a graph G we give a polynomial algorithm for an optimal coloring by delivering a perfect order on its nodes. This class of graphs is shown to be different from the known classes of perfectly orderable graphs. © 1988 Academic Press, Inc.

INTRODUCTION

Chvátal [2] has introduced the class of *perfectly orderable graphs.* It consists of graphs for which an ordering of the nodes can be found such that for any subgraph G' of G, the sequential node coloring algorithm based on this induced order in G' ("always use the smallest possible color") gives an optimal coloring for G'.

An ordering of the nodes of G has this property iff it induces no *obstruction.* An obstruction is an induced $P_4(a, b, c, d)$ (path on 4 nodes) consisting of edges $[a, b]$, $[b, c]$, and $[c, d]$ with $a<b$ and $c>d$ [2]. Such orderings are called *perfect orderings.* Our purpose is to consider a construction which associates with a graph G another graph G' which delivers a perfect order on G if G' is bipartite.

$P_k(x_1, x_2, ..., x_k)$ will denote a path on k nodes consisting of edges $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{k-1}, x_k]$. $\bar{P}_k(x_1, x_2, ..., x_k)$ will denote its complement. A pair of nonadjacent nodes a, b will be denoted $\overline{[a, b]}$. $[x_1, x_2, ..., x_{k-1}, x_k]$ will denote the set of edges $\{[x_1, x_2], [x_2, x_3], ..., [x_{k-1}, x_k]\}$. All graph theoretical terms not defined here can be found in Berge [1].

Basis Ideas

Our purpose is to construct a graph G'. A choice of nodes in this new graph will provide an orientation and a perfect order on G. The first idea is

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to consider as node in G' every unordered pair (x, z) for which we have an induced $P_3(x, y, z)$ in G. The second idea is to choose a set of nodes in G'. For every node (x, z) that we choose, we consider all $P_3(x, y, z)$ in G:

- if $\exists P_4(x, y, z, w)$ in G, the edge $[x, y]$ will be oriented $x \leftarrow y$.

The choice of nodes in G' must satisfy several constraints if we want to be sure that it will provide a perfect order on G. These constraints will give us suggestions for defining the edges in G'.

Constraints

We must avoid obstructions. For every $P_4(a, b, c, d)$ we know that there will be nodes (a, c) and (b, d) in G':

- if we choose the node (a, c), the edge $[a, b]$ will be oriented $a \leftarrow b$
- if we choose the node (b, d), the edge $[c, d]$ will be oriented $c \rightarrow d$.

In order to avoid obstructions, we must choose *at least* one of them.

Every edge can receive only one orientation. Let us consider the graph $\bar{P}_5(a, b, c, d, e)$. (a, b) and (d, e) will be nodes in G':

- if we choose the node (a, b), the edge $[b, d]$ will be oriented $b \leftarrow d$
- if we choose the node (d, e), the edge $[b, d]$ will be oriented $b \rightarrow d$.

We must choose *at most* one of them.

The last idea now is to choose exactly one of the 2 possible nodes for both constraints:

$$\left.\begin{array}{ll} \text{obstructions:} & \text{at least } 1 \\ \text{orientation:} & \text{at most } 1 \end{array}\right\| \Rightarrow \text{we choose exactly } 1.$$

CONSTRUCTION

Let us consider the construction which associates with a graph $G = (V, E)$ another graph $G' = (V', E')$ where

$$V' = \{(x, z)/x, z \in V, [x, z] \notin E, \exists y \text{ such that } \{[x, y], [y, z]\} \subseteq E\}$$

$$E' = \{[(x, y), (z, w)]/P_4(x, z, y, w) \text{ or } \exists r \text{ such that } \overline{P_5}(x, y, r, z, w)\}.$$

Figure 1 shows an example of this construction.

Let us suppose now that G' is bipartite; by choosing one of its parts, we satisfy the 2 constraints.

DEFINITION. G is called a *bipartable graph* if G' is bipartite.

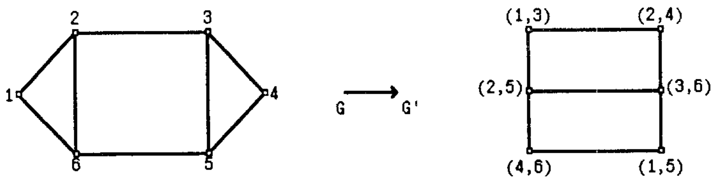


FIGURE 1

THE ALGORITHM

Let G be a bipartable graph. We give an algorithm which determines a perfect order on G. (Figure 2 shows an application of this algorithm)

(a) Construct G'.

(b) Choose 1 part V_1 of the bipartite graph G'. For every node $v=(x, z)$ of V_1, consider all nodes y of V such that $P_3(x, y, z)$ in G. The edge $[x, y]$ is oriented $x \leftarrow y$ if $\exists\ P_4(x, y, z, w)$ in G.

(c) Choose a total order on V such that $b<a$ (b before a) if $b \rightarrow a$.

The proof of the validity of this algorithm will be divided in 3 parts:

(1) we do not create obstructions

(2) every edge can receive only 1 orientation

(3) we do not create circuits.

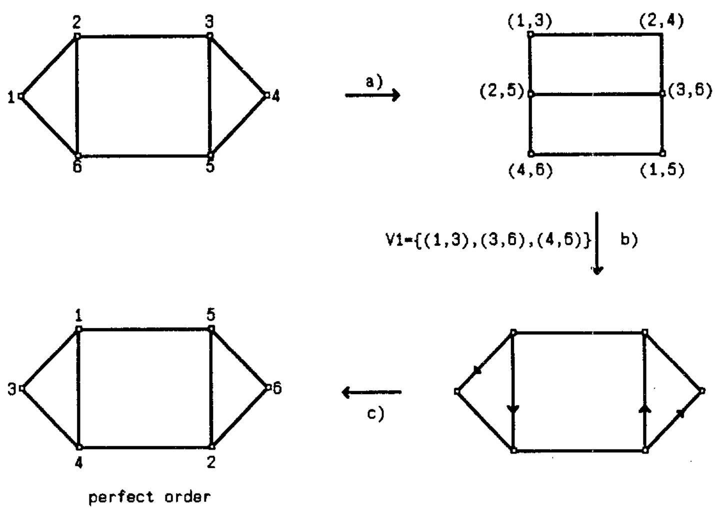


FIGURE 2

PART 1. *We do no create obstructions.*

Proof. For every $P_4(a, b, c, d)$ we have $[(a\ c), (b, d)]$ in E'. Since either (a, c) or (b, d) is in V_1, we have $b \to a$ or $c \to d$ and so $b < a$ or $c < d$. We cannot have obstructions. ▮

PART 2. *Every edge can receive only* 1 *orientation.*

Proof. Let us consider that an edge $[a, b]$ in E received 2 opposite orientations. We know that:

– a node (a, c) was in V_1 and gave $b \to a$ because of a $P_4(a, b, c, d)$
– a node (b, e) was in V_1 and gave $b \leftarrow a$ because of a $P_4(b, a, e, f)$
if $f = d$: with or without the edge $[e, c]$ in E, we have $[(a, c), (e, b)]$ in E'
if $f \neq d$: in order to avoid $[(a, c), (e, b)]$ in E', we have $[e, c]$, $\overline{[d, e]}$, $\overline{[f, c]}$ in E
– if we have $\overline{[f, d]}$ in E, we have $[(e, b), (f, c), (e, d), (a, c)]$ in E'
– if we have $[f, d]$ in E, we have $[(e, b), (f, c), (b, d), (a, c)]$ in E'.

For all these cases, (a, c) and (b, e) cannot both be in V_1. ▮

PART 3. *We do not create circuits.*

LEMMA 1. *We have no circuits of length* 3.

Let us suppose that we have a circuit $1 \leftarrow 2 \leftarrow 3$. We must have nodes 4, 5, 6 in V, edges $[4, 2]$, $\overline{[4, 1]}$, $[5, 3]$, $\overline{[5, 2]}$, $[6, 1]$, $\overline{[6, 3]}$ in E, and nodes $(1, 4)$, $(2, 5)$, $(3, 6)$ in V_1. Note that 1, 2, 3, 4, 5, 6 are all distinct.
– $1 \leftarrow 2$ is the result of the node $(1, 4)$ in V_1 and a $P_4(1, 2, 4, 7)$. Nodes 7 and 5 are different since otherwise we would have $[(1, 4), (2, 5)]$ in E'.

– Let us suppose we have not the edge $[4, 3]$ in E. We would have
– $[1, 5]$ and $[4, 5]$ in E (otherwise $[(1, 4), (2, 7), (3, 4), (2, 5)]$ in E')
– $\overline{[5, 7]}$ in E (otherwise $2 \leftarrow 1$ because of $P_4(2, 1, 5, 7)$)
– $[3, 7]$ in E (otherwise $3 \leftarrow 2$ because of $P_4(3, 2, 4, 7)$)
– a node 8 in V so that $(2, 5)$ in V_1 and P_4 $(2, 3, 5, 8)$ gave $3 \to 2$ ($8 \neq 6$ otherwise we would have $[(2, 5), (3, 6)]$ in E')
– $\overline{[4, 8]}$ in E (otherwise $[(2, 5), (4, 3), (2, 7), (1, 4)]$ in E')
– $\overline{[1, 8]}$ in E (otherwise $[(1, 4), (2, 5)]$ in E')
– $2 \leftarrow 1$ because of $P_4(2, 1, 5, 8)$. *Contradiction.*

So we know that we have the edge $[4, 3]$. By symmetry, we also have $[5, 1]$ and $[6, 2]$ in E.

- Let us suppose we have the edge $[4, 5]$ in E. We would have
  - $\overline{[5, 7]}$ in E (otherwise $[(1, 4), (2, 5)]$ in E')
  - $[3, 7]$ in E (otherwise $1 \leftarrow 3$ because of $P_4(1, 3, 4, 7)$)
  - a node 8 ($\neq 6$) as before
  - $\overline{[7, 8]}$ in E (otherwise $[(2, 7), (3, 8)]$ in E')
  - $[(1, 4), (5, 7), (3, 8), (2, 5)]$ in E'. *Contradiction.*

So we know that we have not the edge $[4, 5]$. By symmetry we have not $[4, 6]$ and $[5, 6]$ in E.

We now have $[(6, 3), (2, 5), (1, 4), (6, 3)]$ which is an odd cycle in G'. This contradicts the fact that the graph G was bipartable.

#

LEMMA 2. *The* $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ *cannot be a part of a minimal circuit (a minimal circuit is a circuit without shortcut).*

Let us suppose $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ is a part of a minimal circuit.

Claim 1. The $1 \leftarrow 2$ does not result from $(1, 3)$ in V_1 and a $P_4(1, 2, 3, x)$ in G.

Case 1.1. The $1 \leftarrow 2$ results from $P_4(1, 2, 3, 5)$; $2 \leftarrow 3$ results from $P_4(2, 3, 4, 6)$. We would have:

- $[1, 4]$ and $\overline{[4, 5]}$ in E (otherwise $[(1, 3), (2, 4)]$ in E')
- $1 \leftarrow 4$ because of $P_4(1, 4, 3, 5)$. *Contradiction,* the circuit was not minimal.

Case 1.2. The $1 \leftarrow 2$ results from $P_4(1, 2, 3, 5)$; $2 \leftarrow 3$ results from $P_4(2, 3, 6, 7)$; we have $[1, 4]$ in E. We would have:

- $\overline{[5, 6]}$, $\overline{[1, 7]}$, and $[1, 6]$ in E (otherwise $[(1, 3), (2, 6)]$ in E')
- $[5, 7]$ and $\overline{[4, 5]}$ in E (otherwise $[(1, 3), (5, 6), (3, 7), (2, 6)]$ in E')
- $1 \leftarrow 4$ because of $P_4(1, 4, 3, 5)$. *Contadiction,* the circuit was not minimal.

Case 1.3. The $1 \leftarrow 2$ results from $P_4(1, 2, 3, 5)$; $2 \leftarrow 3$ results from $P_4(2, 3, 6, 7)$; we have $\overline{[1, 4]}$ in E. We would have:

- $[1, 6]$ and $\overline{[1, 7]}$ in E (otherwise $[(1, 3), (2, 6)]$ in E')
- let us suppose we have not the edge $[2, 4]$ in E. We would have:
  - $\overline{[4, 6]}$ in E (otherwise $[(1, 3), (2, 6)]$ in E)
  - $\overline{[4, 7]}$ in E (otherwise $[(1, 3), (2, 4), (3, 7), (2, 6)]$ in E')
  - $[(1, 3), (4, 6), (3, 7), (2, 6)]$ in E'. *Contradiction* so we know that we have the edge $[2, 4]$ in E (and that $5 \neq 4$).
- $\overline{[5, 6]}$ in E (otherwise $[(1, 3), (2, 6)]$ in E')
- $[5, 7]$ in E otherwise $[(1, 3), (5, 6), (3, 7), (2, 6)]$ in E')
- $[(1, 3), (2, 5), (3, 7), (2, 6)]$ in E'. *Contradiction.*

By this first claim, we also know that $2 \leftarrow 3$ does not result from $(2, 4)$ in V_1 and a $P_4(2, 3, 4, x)$ in G.

Claim 2. The $1 \leftarrow 2$ does not result from $(1, 4)$ in V_1 and a $P_4(1, 2, 4, x)$ in G.

Suppose $1 \leftarrow 2$ results from $P_4(1, 2, 4, 7)$; $2 \leftarrow 3$ results from $P_4(2, 3, 5, 6)$; and $3 \leftarrow 4$ results from $P_4(3, 4, 8, 9)$.
We would have:

– Let us suppose we have the edge $[1, 5]$ in E. We would have:
 – $[4, 5]$ in E (otherwise $[(1, 4), (2, 5)]$ in E')
 – $[1, 6]$ in E (otherwise $2 \leftarrow 1$ because of $P_4(2, 1, 5, 6)$)
 – $[4, 6]$ in E (otherwise $[(1, 4), (2, 5)]$ in E')
 – $[1, 3]$ in E (otherwise $[(2, 5), (1, 3), (2, 6), (1, 4)]$ in E')
 – $[3, 7]$ in E (otherwise $1 \leftarrow 3$ because of $P_4(1, 3, 4, 7)$)
 – $\overline{[6, 7]}$ in E (otherwise $[(1, 4), (2, 7), (3, 6), (2, 5)]$ in E')
 – $[7, 8]$ and $[2, 8]$ in E (otherwise $[(3, 8), (1, 7), (3, 6), (2, 5)]$ in E')
 – $\overline{[5, 7]}$ in E (otherwise $[(1, 4), (2, 5)]$ in E')
 – $[5, 8]$ in E (otherwise $[(3, 8), (5, 7), (3, 6), (2, 5)]$ in E')
 – $\overline{[6, 8)}$ in E (otherwise $[(3, 8), (1, 7), (3, 6), (2, 5)]$ in E')
 – $[(1, 4), (6, 8), (5, 7), (1, 4)]$ in E'. *Contradiction*, G was not bipartable so we know that we have not the edge $[1, 5]$ in E.

– $\overline{[4, 5]}$ in E (otherwise $[(1, 4), (2, 5)]$ in E')
– $[4, 6]$ in E (otherwise $4 \leftarrow 3$ because of $P_4(4, 3, 5, 6)$)
– Let us suppose we have the edge $[1, 6]$ in E. We would have:
 – $[1, 3]$ in E (otherwise $[(1, 3), (2, 5), (3, 6), (1, 5), (2, 6), (1, 3)]$ in E')
 – $\overline{[6, 7]}$, $[7, 8]$, $[2, 8]$, and $\overline{[6, 8]}$ as before
 – $[5, 8]$ in E (otherwise $[(3, 8), (1, 5), (3, 6), (2, 5)]$ in E')
 – $[2, 9]$, $[7, 9]$ and $[6, 9]$ in E (otherwise $[(3, 8), (4, 9), (6, 8), (1, 4)]$ in E')
 – $[(2, 5), (3, 6), (1, 5), (2, 6), (4, 9), (3, 8)]$ in E'. *Contradiction* so we know that we have not the edge $[1, 6]$ in E.

– $\overline{[1, 3]}$ in E (otherwise $1 \leftarrow 3$ because of $P_4(1, 3, 5, 6)$)
– $\overline{[1, 8]}$ in E (otherwise $[(1, 4), (3, 8)]$ in E')
– $[5, 8]$ and $[2, 8]$ in E (otherwise $[(3, 8), (4, 5), (3, 6), (2, 5)]$ in E')
– $\overline{[6, 8]}$ in E (otherwise $[(2, 5), (3, 8)]$ in E')
– $[6, 9]$ and $[2, 9]$ in E (otherwise $[(3, 8), (4, 9), (6, 8), (4, 5), (3, 6), (2, 5)]$ in E')
– $[(3, 8), (4, 9), (2, 6), (1, 4)]$ in E'. *Contradiction.*

So we know that $1 \leftarrow 2$ results from a $P_4(1, 2, 5, 10)$ and $2 \leftarrow 3$ from a $P_4(2, 3, 6, 8)$, where 1, 2, 3, 4, 5, 6, 8 are all different nodes.

Claim 3. The $3 \leftarrow 4$ results from $(3, x)$ in V_1, where $x \neq 1, 5$.

Case 3.1. The $3 \leftarrow 4$ results from $(3, 1)$ in V_1 and a $P_4(3, 4, 1, 7)$ in G. We would have:

- $[1, 6]$ and $\overline{[1, 8]}$ in E (otherwise $[(1, 3), (2, 6)]$ in E')
- $2 \leftarrow 1$ because of $P_4(2, 1, 6, 8)$. *Contradiction.*

Case 3.2. The $3 \leftarrow 4$ results from $(3, 5)$ in V_1 and $P_4(3, 4, 5, y)$ in G. We would have:

- $[5, 6]$ and $\overline{[5, 8]}$ in E (otherwise $[(2, 6), (3, 5)]$ in E')
- $[1, 6]$ and $\overline{[1, 8]}$ in E (otherwise $[(1, 5), (2, 6)]$ in E')
- $2 \leftarrow 1$ because of $P_4(2, 1, 6, 8)$. *Contradiction.*

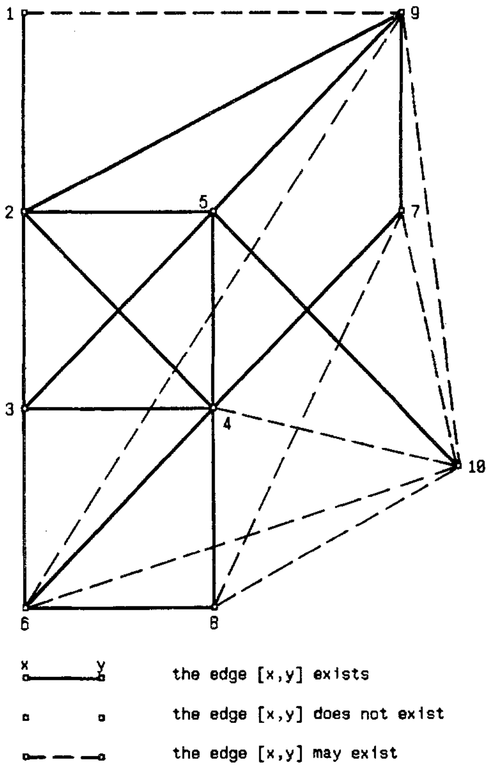
So we know that

$1 \leftarrow 2$ results from a $P_4(1, 2, 5, 10)$
$2 \leftarrow 3$ results from a $P_4(2, 3, 6, 8)$
$3 \leftarrow 4$ results from a $P_4(3, 4, 7, 9)$.

Claim 4. $H(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ is an induced configuration of G (see Fig. 3).

Let us suppose we have no $H(\ )$ *as induced configuration*:

- Let us suppose we have not the edge $[4, 6]$ in E. We would have:
  - $[4, 8]$ in E (otherwise $[(2, 6), (3, 8), (4, 6)]$ in E', (4.6) in V_1, $3 \rightarrow 4$)
  - let us suppose we have the edge $[2, 4]$ in E. We would have:
    - $[6, 7]$, $[2, 7]$, and $\overline{[6, 9]}$ in E (otherwise $[(2, 6), (3, 8), (4, 6), (3, 7)]$ in E')
    - $\overline{[7, 8]}$ and $\overline{[2, 9]}$ in E (otherwise $[(2, 6), (3, 7)]$ in E')
    - $[(2, 6), (7, 8), (4, 9), (3, 7)]$ in E'. *Contradiction* so we know that we have not the edge $[2, 4]$ in E
  - $[2, 7]$ and $\overline{[8, 7]}$ in E (otherwise $[(3, 7), (2, 4), (3, 8), (2, 6)]$ in E')
  - $[6, 7]$ and $\overline{[6, 9]}$ in E (otherwise $[(2, 6), (3, 7)]$ in E')
  - $[8, 9]$ and $\overline{[2, 9]}$ in E (otherwise $[(3, 7), (4, 9), (7, 8), ((2, 6)]$ in E')
  - $[(3, 7), (2, 9), (7, 8), (2, 6)]$ in E'. *Contradiction* so we know that we have the edge $[4, 6]$ in E. By symmetry, we also have $[3, 5]$
- Let us suppose we have the edge $[6, 7]$ in E. We would have:
  - $[2, 7]$ and $\overline{[8, 7]}$ in E (otherwise $[(2, 6), (3, 7)]$ in E')
  - let us suppose we have not the edge $[6, 9]$ in E. We would have:



CONFIGURATION H(1,2,3,4,5,6,7,8,9,10)

FIGURE 3

– [8, 9] in E (otherwise [(3, 7), (6, 9), (7, 8), (2, 6)] in E')

– [(3, 7), (6, 9), (3, 8), (2, 6)] in E'. *Contradiction* so we know that we have the edge [6, 9]

– $\overline{[2, 9]}$ in E (otherwise [(2, 6), (3, 7)] in E')

– let us suppose we have the edge [8, 9] in E. We would have:

– [4, 8] in E (otherwise [(3, 7), (4, 9), (7, 8), (2, 6)] in E')

– [(3, 7), (4, 9), (3, 8), (2, 6)] in E'. *Contradiction* so we know that we have not the edge [8, 9] in E

– $\overline{[4, 8]}$ in E (otherwise [(3, 7), (4, 9), (7, 8), (2, 6)] in E')

– $\overline{[2, 4]}$ in E (otherwise $2 \leftarrow 4$ because of $P_4(2, 4, 6, 8)$)

– [(3, 7), (2, 4), (3, 9), (2, 6)] in E'. *Contradiction* so we know that we have not the edge [6, 7] in E. By symmetry, we also have $\overline{[5, 6]}$ in E. But now we have:

- $\overline{[2, 7]}$ in E (otherwise $[(2, 6), (3, 7)]$ in E')
- $\overline{[1, 6]}$ in E (otherwise $[(1, 5), (2, 6)]$ in E')
- $\overline{[5, 7]}$ in E (otherwise $[(2, 6), (3, 8), (5, 6), (3, 7)]$ in E')
- Let us suppose we have the edge $[2, 4]$ in E. We would have:
    - $[2, 9]$ and $[4, 8]$ in E (otherwise $2 \leftarrow 4$ because of $P_4(2, 4, 7, 9)$ and $P_4(2, 4, 6, 8)$)
    - $\overline{[8, 9]}$ in E (otherwise $[(2, 6), (3, 8), (4, 9), (3, 7)]$ in E')
    - $\overline{[5, 8]}$ in E (otherwise $[(2, 6), (3, 8), (5, 6), (2, 8), (4, 9), (3, 7)]$ in E')
    - $\overline{[1, 8]}$ in E (otherwise $[(1, 5), (2, 8), (4, 9), (3, 7)]$ in E')
    - $[4, 5]$ in E (otherwise $[(3, 7), (4, 5), (3, 8), (2, 6)]$ in E')
    - $\overline{[1, 3]}$ in E (otherwise $[(2, 6), (3, 8), (1, 6)]$ in E', $(1, 6)$ in V_1, $1 \leftarrow 3$)
    - $\overline{[1, 7]}$ in E (otherwise $[(1, 5), (2, 7), (1, 3), (2, 6)]$ in E')
    - Let us suppose we have the edge $[1, 4]$. We would have:
        - $[4, 10]$ in E (otherwise $1 \leftarrow 4$ because of $P_4(1, 4, 5, 10)$)
        - $[(1, 5), (2,10), (4, 9), (3, 7)]$ in E'. *Contradiction* so we know that we have not the edge $[1, 4]$ in E.
    - $[5, 9]$ in E (otherwise $[(1, 5), (2, 10), (5, 9), (2, 7), (1, 4), (2, 6)]$ in E')
    - configuration $H(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$. *Contradiction* so we know that we have not the edge $[2, 4]$ in E. By symmetry, we have not $[1, 3]$ in E. But we have now $[(3, 7), (2, 4), (1, 3), (2, 6)]$ in E'. *Contradiction.*

We can conclude that if we have $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ as part of a minimal circuit, we also have the induced configuration $H(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

Claim 5. The $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ is not a part of a minimal circuit.

Let us suppose it is is a part of a minimal circuit. We would have:

- configuration $H(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$
- a node 11 in V such that $11 \leftarrow 1$
    - $11 \neq 2$ since $2 \rightarrow 1$
    - $11 \neq 3, 4, 5, 6, 7, 8, 10$ since $[11, 1]$
    - $11 \neq 9$ since $1 \leftarrow 9$ $([1, 7), (4, 9), (3, 7)]$ in E', $(1, 7)$ in V_1)

$11 \leftarrow 1$ results from a $P_4(11, 1, 12, x)$

- $12 \neq 2$ because of Claim 1
- $12 \neq 3, 4, 5, 6, 7, 8, 10$ since $[12, 1]$
- $12 \neq 9$ since $\overline{[5, 12]}$ (by symmetry with $\overline{[6, 7]}$)
- $\overline{[5, 11]}$ in E (otherwise $[(11, 12), (1, 5)]$ in E')
- $[2, 11]$ in E (otherwise $[(1, 5), (2, 11), (1, 3), (2, 6)]$ in E')

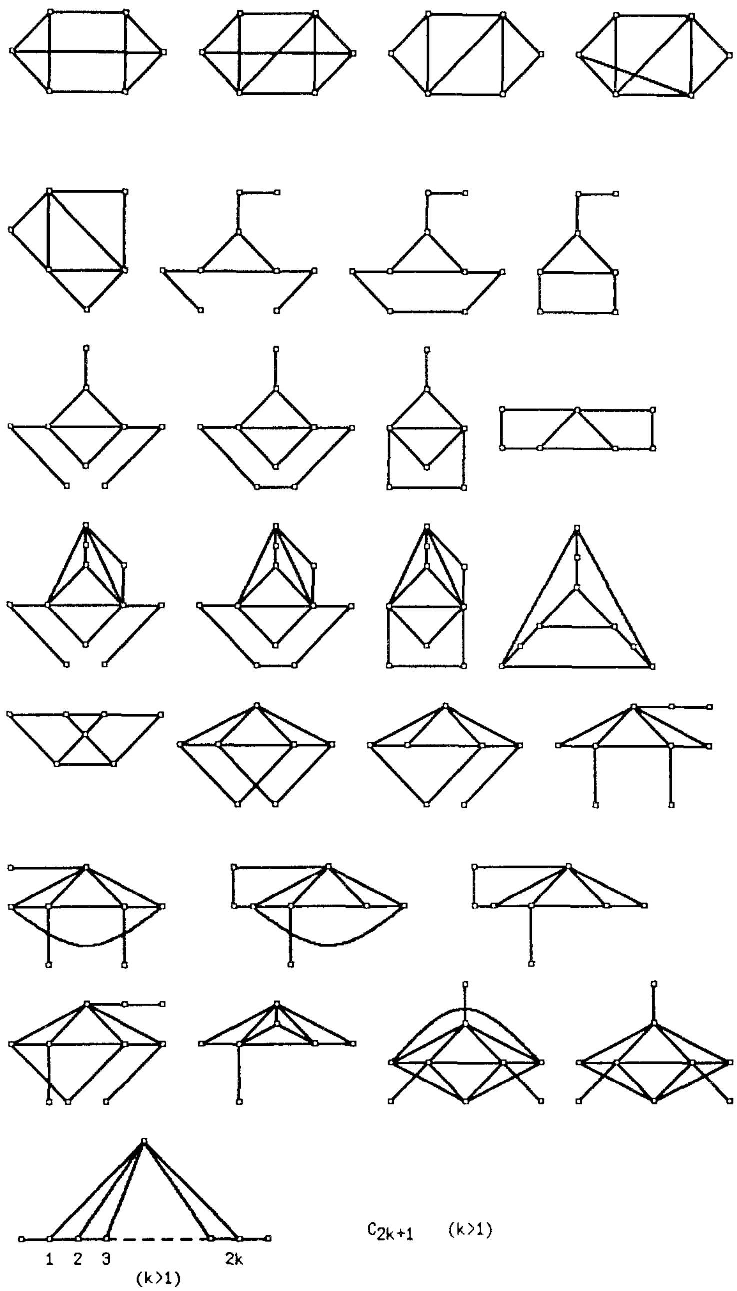


FIGURE 4

– [10, 11] in E (otherwise [(1, 5), (2, 10), (5, 11)] in E', (5, 11) in V_1, $11 \leftarrow 2$)

– [6, 11] in E (otherwise [(2, 6), (5, 11), (3, 10), (1, 5)] in E')

– [3, 11] in E (otherwise [(2, 6), (3, 11), (2, 10), (1, 5)] in E')

– [4, 11] in E (otherwise [(2, 6), (4, 11), (1, 3), (2, 6)] in E', G non-bipartable)

– [8, 11] in E (otherwise [(2, 6), (3, 8), (6, 11)] in E', (6, 11) in V_1, $11 \leftarrow 3$)

– [(3, 8), (6, 11), (2, 8), (1, 4), (2, 6), (3, 8)] in E', G non-bipartable. *Contradiction.*

#

FINAL REMARKS

1. Since the construction of G' and the test of whether G' is a bipartite graph or not can be computed in polynomial time, recognition and coloring of a bipartable graph can be done in polynomial time. Figure 4 shows some minimal non-bipartable graphs.

2. Let us now consider 7 prototypes of perfectly orderable graphs: (a) comparability graphs, (b) interval graphs, (c) triangulated graphs, (d) Welsh–Powell perfect graphs [4], (e) complements of triangulated graphs, (f) brittle graphs [3], and (g) complements of tolerance graphs [5].

F1 (resp. F2, F3, F4, F4, F4, F4) is a graph of class (a) (resp. (b), (c), (d), (e), (f), (g)) which is not bipartable (see Fig. 5)).

$\overline{F4}$ (resp. C4, C6, C6, 2K2, C6, C6) is a bipartable graph but not of class (a) (resp. (b), (c), (d), (e), (f), (g)) (see Fig. 5).

We can conclude that the class of bipartable graphs is different from other classes of perfectly orderable graphs.

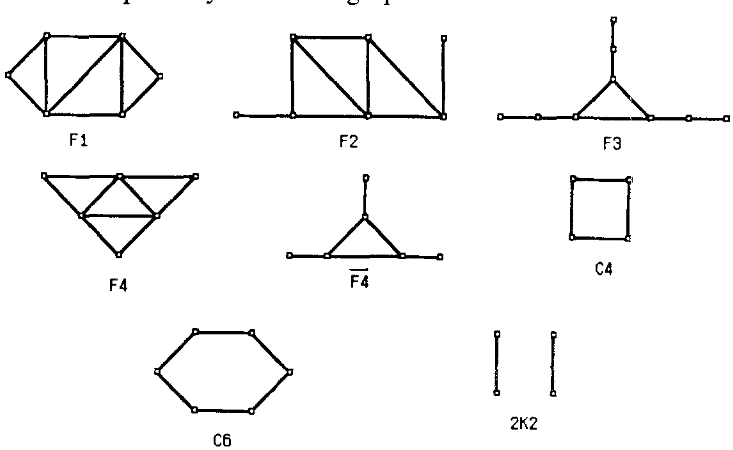


FIGURE 5

Note added in proof. After this paper was written, a referee pointed out that the results presented here generalize Theorem 4.1 in P. L. Hammer and N. V. R. Mahadev, "Bithreshold Graphs", *SIAM J. Algebraic and Discrete Methods* **6** (1985), 497–506: intersections of two threshold graphs are bipartable.

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